Constellation Models
Single Object, One Feature

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1 Introduction

We derive the method for deciding whether the target object is in the image or not, for the case of a single object modeled by a single feature. We do not address the labeling problem here.

2 Notation

The variable \( O \) denotes the presence of the object. It can take on values \( O_1 \) (object there) or \( O_0 \) (object absent).

\[ D \in \{ D_0, D_1 \} \] denotes the fact that the object is detected.

\( N \) is the number of observed points.

\( x \) is a vector of candidate positions.

Labels are denoted by \( l \), where \( l_0 \) is the zero-label and \( l_i \) has a 1 in entry \( i \) and zeros otherwise.

\( A \) is the size of the image (length for 1D, area for 2D).

3 Solution

The world (at least everything that we observe in one image), can be described through the following joint density: \( p(x, l, N, D, O) \).

The question we want to answer can be stated as: “Given the observed data, is the object in the image.” Note the emphasis on observed data. There is also unobserved data, which we will have to eliminate from the joint density. We base our decision on the posterior probability of the presence of the object, which can be rewritten using Baye’s rule:

\[
p(O|x, N) = \frac{p(x, N|O)p(O)}{p(x)}
\]

This reflects our question, since \( x \) and \( N \) are observed, while \( l \) and \( D \) are not. Note that we are not concerned about \( p(x) \) since this term will cancel out when comparing classes. We now write out the class-conditional density, \( p(x, N|O) \) and integrate (sum) over the unobserved variables.

\[
p(x, N|O) = \sum_{l, D} p(x, l, N, D|O).
\]
We can factorize this density as follows,

\[ p(x, N|O) = \sum_{l \in D} p(x|N, l, D, O) p(l|N, D, O) p(N|D, O) p(D|O). \]

Note that we do not need to sum over all possible combinations of labels and values of \( D \), since \( p(l|N, D, O) \) is only different from zero, if \( l, D \) and \( N \) are consistent. This means that, for the case where \( D = D_0 \), only the zero-label needs to be considered, whereas in the case of \( D = D_1 \) only labels of length \( N \), containing exactly one "1" (i.e. labels \( L_1 \) through \( L_N \)) need to be considered. We can therefore do the sum over \( D \) explicitly and obtain

\[ p(x, N|O) = p(x, N, D_0, L_0|O) + \sum_{i=1}^{N} p(x|N, L_i, D_1, O) p(L_i|N, D_1, O) p(N|D_1, O) p(D_1|O). \]

We now consider all the factors of the density inside the summation. The first factor, \( p(x|N, L_i, D_1, O) \), represents the density over the feature positions. We need to evaluate the feature which is assigned the foreground label under the foreground density, \( p_{fg}(x) \), and all remaining features under the background density which is uniform \((p_{bg}(x) = 1/\lambda)\). We obtain

\[ p(x|N, L_i, D_1, O) = p_{fg}(x(i)) \frac{1}{\lambda^{N-1}}. \]

The second factor, \( p(L_i|N, D_1, O) \), is simply equal to \( 1/N \), since there are \( N \) different labels over which we are summing.\(^1\)

Finally, \( p(N|D_1, O) \) is equal to \( P_{\text{Pois}}(N-1) \). This is because, given that the object has been detected, exactly \( N-1 \) points have to be noise. Putting everything together, we obtain:

\[ p(x, N|O) = \frac{1}{\lambda N} P_{\text{Pois}}(N) p(D_0|O) + \sum_{i} [p_{fg}(x(i))] \frac{1}{\lambda^{N-1}} \frac{1}{N} P_{\text{Pois}}(N-1) p(D_1|O) \]

Since we want to compare the posterior probability of the object being in the image with that of the object not being in the image, we can make a decision by thresholding the following likelihood ratio, where we use the reasonable assumption that \( p(D_1|O_0) = 0 \) and, therefore, \( p(D_0|O_0) = 1 \):

\[ \frac{p(x, N|O_1)}{p(x, N|O_0)} = \frac{p(D_0|O_1)}{\lambda N} P_{\text{Pois}}(N) + \frac{p(D_1|O_1)}{\lambda^{N-1}} \frac{1}{N} P_{\text{Pois}}(N-1) \sum_{i} p_{fg}(x(i)). \]

This can be simplified to

\[ \frac{p(x, N|O_1)}{p(x, N|O_0)} = \frac{p(D_0|O_1)}{\lambda} + \frac{p(D_1|O_1)}{\lambda} \frac{1}{N} P_{\text{Pois}}(N-1) \sum_{i} p_{fg}(x(i)) \]

\[ = \frac{p(D_0|O_1)}{\lambda} + \frac{p(D_1|O_1)}{\lambda} \frac{1}{N} \sum_{i} p_{fg}(x(i)) \]

Here, \( \lambda \) is the parameter of the Poisson distribution.

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\(^1\)Since we assume that our data are put into the vector \( x \) in random order (shuffled), we do not need to be concerned about certain labels occurring more often than others.
We can validate this result by assuming that $p_{10}$ is also a uniform density. In this case we obtain

$$\frac{p(x, N|O_1)}{p(x, N|O_0)} = p(D_0|O_1) + p(D_1|O_1) \frac{N}{\lambda},$$

which is intuitively appealing, since, if we furthermore assume that the object is always detected when present, this expression tells us to decide that the object is present whenever we have more detections than the average number.