Lecture 2: Matched filtering

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1 Detecting a signal in white noise when the signal is known

Linear operator: matched filter k is best. Response R_n to Gaussian white noise of mean $\mu = 0$ and std σ is:

$$\begin{aligned} R_n &= < k, n >= \sum_i k_i n_i \\ \mathrm{E}R_n &= \sum_i k_i \mathrm{E}n_i = 0 \\ \mathrm{E}R_n^2 &= \sum_{i,j} k_i k_j \mathrm{E}n_i n_j = \sum_i k_i^2 \mathrm{E}n_i^2 = \sigma^2 \|k\|^2 \end{aligned} \tag{1}$$

The response to the signal is:

$$R_s = < k, s > \tag{2}$$

Notice that this response is proportional to the norm of the kernel k, the same way the variance of the response in noise is. We may as well set $\|k\| = 1$ from now on.

The question is then how to maximize R_s with respect to k ($\mathrm{E}R_n$ depends only on the norm of k which is set to 1). This is quite clearly maximized when

$$k = \frac{s}{\|s\|}$$

This may be verified either using constrained maximization (e.g. using Lagrange multipliers) or simply noting that in order to maximize the product of s and k one needs to align k with s.

What do we do when the position is not known? Use the product at all locations and consider the local maxima. This is called a correlation integral:

$$R(x) = s * k =< s(y-x), k >= \int s(y-x) k(y) dy$$

See `Matlab` example.

What when there are two possible signals? Use 2 filters and pick the local maximum that is largest.

2 Variable signal

What when the signal is variable (eg. lighting, deformations, translation)? For example consider an object with Lambertian reflectance....

Suppose that one could only use one kernel, then one would pick the kernel that maximizes the average product with the observed signals s_i:

$$\begin{aligned} k &= \arg\max_k \sum_i < k, s_i >^2 = k^T S S^T k \\ \text{where } S &= [s_1 \ldots s_N] \end{aligned}$$

(here, for simplicity of notation, we think of both k and the s_i as column vectors).

Now notice that $A = SS^T$ is a positive semidefinite symmetric matrix. Therefore its eigenvalues are real and semipositive and its eigenvectors are an orthonormal set. Call u_i the set of eigenvectors of A (equivalently, they are the left-eigenvectors of S, and call σ_i the corresponding eigenvalues. Suppose that they are sorted in descending order of magnitude. Then A may be expressed in terms of the basis formed by the u_i and the maximization has a simple solution:

$$\begin{aligned} A &= SS^T = \sum_i \sigma_i u_i u_i^T \\ k &= \arg\max_k k^T A k = u_1 \\ \text{and} & \quad \max_k k^T A k = \sigma_1 ||k||^2 = \sigma_1 \end{aligned}$$

Suppose that we allowed k to belong to a r-dimensional space with $r << N$. Then the solution would be $k_i = u_i$. This technique is called ‘principal component analysis’. Use the `svd` function in `Matlab` to calculate the principal components as demonstrated in the code `lecture2.m`.